ON IDENTIFIABILITY FOR MULTIDIMENSIONAL ARMAX MODEL*†

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Abstract

This paper gives a definition of identifiability for multidimensional linear input-output systems and presents a necessary and sufficient condition for its satisfaction. For a class of identifiable systems it is also shown that the unknown coefficients of the system can consistently be estimated by a recursive algorithm.

1. Introduction

The basic idea of identifiability is the possibility of determining a system or its parameters from the input-output data. Several different definitions of identifiability are given in the survey paper [1] for one-dimensional systems. However, from the following example we shall see that the situation for multidimensional systems is quite different from the one-dimensional case.

Let the linear input-output system be described by

$$A(z)y_t = B(z)u_t, \tag{1.1}$$

where $A(z)$ and $B(z)$ are polynomials in the shift-back operator z.

If both y_t and u_t are one-dimensional, then coprimeness of $A(z)$ and $B(z)$ is necessary and sufficient for uniquely determining parameters of $A(z)$ and $B(z)$ from the data. But in the multidimensional case the left-coprimeness of $A(z)$ and $B(z)$ does not guarantee the uniqueness of representation (1.1).

Example 1.1. Let

$$A(z) = I + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z + Iz^2, \quad B(z) = Iz + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z^2.$$

They are left-coprime, since

$$A(z)M(z) + B(z)N(z) = I,$$

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where

$$M(z) = I - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z, \qquad N(z) = -Iz + 2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z^2.$$

If we multiply $A(z)$ and $B(z)$ from the left by $I - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z$, then the system turns to

$$A'(z)y_t = B'(z)u_t, \tag{1.2}$$

where

$$A'(z) = I + Iz^2 - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z^3, \qquad B'(z) = Iz.$$

It is easy to see that $A'(z)$ and $B'(z)$ also are left-coprime. Thus the input-output data cannot uniquely define parameters of the system.

In this paper, we give a definition of identifiability for multidimensional linear systems and present a necessary and sufficient condition for identifiability. When this condition is satisfied, strongly consistent estimates for the unknown coefficients are derived.

2. Identifiability and Identification Methods

We now consider the system described by an ARMAX model:

$$\begin{aligned} A(z)y_t &= B(z)u_t + C(z)w_t, \qquad t > 0; \\ y_t &= w_t = 0, \quad u_t = 0, \qquad t \le 0, \end{aligned} \tag{2.1}$$

where y_t, u_t and w_t are m-output, n-input and m-driven noise, respectively; $A(z)$, $B(z)$ and $C(z)$ are given by the following equations:

$$A(z) \triangleq I + A_1 z + \cdots + A_p z^p, \qquad p \ge 0, \tag{2.2}$$

$$B(z) \triangleq B_1 z + \cdots + B_q z^q, \qquad q \ge 1, \tag{2.3}$$

$$C(z) \triangleq I + C_1 z + \cdots + C_r z^r, \qquad r \ge 0. \tag{2.4}$$

The driven noise $\{w_t, \mathcal{F}_t\}$ is assumed to be a martingale difference sequence with respect to a non-decreasing family of σ-algebras. It is also assumed that

$$\sup_{t \ge 0} E[\, \|w_{t+1}\|^2 \mathcal{F}_t \,] < \infty \qquad \text{a.s.,} \tag{2.5}$$

$$\liminf_{t\to\infty} \frac{1}{t^{1-\varepsilon^*}} \lambda_{\min}\left(\sum_{i=0}^{t} w_i w_i^\tau\right) > 0 \qquad \text{a.s.,} \tag{2.6}$$

$$\limsup_{t\to\infty} \frac{1}{t}\sum_{i=0}^{t} \|w_i\|^2 < \infty \qquad \text{a.s.,} \tag{2.7}$$

where

$$\varepsilon^* = \frac{1}{2\mu + 3}, \qquad \mu = (m+1)p + r + q \tag{2.8}$$

and $\lambda_{\min}(x)$ denotes the minimum eigenvalue of the matrix X.

In recent years there has been made some progress for consistently estimating the unknown coefficient θ

$$\theta^{\tau} = [-A_1 \cdots - A_p \ B_1 \cdots B_q \ C_1 \cdots C_r] \tag{2.9}$$

under various conditions. For example, in Theorem 4 of [2] it is assumed that $A(z)$ is stable, A_p is of row-full-rank and $C^{-1}(z) - \frac{1}{2}I$ is strictly positive real; in [3] it is required that both $C^{-1}(z) - \frac{1}{2}I$ and $C(z) - \frac{\overline{a}}{2}I$ are positive real for some $\overline{a} > 0$, $z^{-1}B(z)$ is stable, A_p is of row-full-rank and $A(z)$, $B(z)$ and $C(z)$ have no common left factor. Obviously, all these conditions are sufficient for identifying θ. Our purpose is to clarify what is the minimum requirement for this.

Definition 2.1. A system described by (2.1) is said to be identifiable if there are no polynomials $A'(z) = I + A'_1 z + \cdots + A'_{p'} z^{p'}$, $B'(z) = B'_1 z + \cdots + B'_{q'} z^{q'}$ and $C'(z) = I + C'_1 z + \cdots + C'_{r'} z^{r'}$ with $p' \le p$, $q' \le q$ and $r' \le r$, respectively, so that $(A'(z))^{-1}B'(z) \equiv (A(z))^{-1}B(z)$ and $(A'(z))^{-1}C'(z) \equiv (A(z))^{-1}C(z)$ unless $A'(z) \equiv A(z)$, $B'(z) \equiv B(z)$ and $C'(z) \equiv C(z)$.

Theorem 2.1. The system (2.1) is identifiable if and only if $A(z)$, $B(z)$ and $C(z)$ have no common left factor and $\operatorname{rank}[A_p \ B_q \ C_r] = m$.

Proof. We first prove the necessity. Assume the system is identifiable. If the converse were true, then there would exist a non-unimodular polynomial matrix $D(z)$ and polynomials $A'(z)$, $B'(z)$ and $C'(z)$ with orders less than or equal to those of $A(z)$, $B(z)$ and $C(z)$, respectively, so that $[A(z) \ B(z) \ C(z)] = D(z)[A'(z) \ B'(z) \ C'(z)]$ which implies $(A'(z))^{-1}B'(z) \equiv (A(z))^{-1}B(z)$ and $(A'(z))^{-1}C'(z) \equiv (A(z))^{-1}C(z)$. Thus by Definition 2.1 we have $A'(z) \equiv A(z)$, $B'(z) \equiv B(z)$ and $C'(z) \equiv C(z)$, and hence $D(z) = I$. The obtained contradiction implies that $A(z)$, $B(z)$ and $C(z)$ have no common left factor.

Further, if $\operatorname{rank}[A_p \ B_q \ C_r] \ne m$ then $\operatorname{rank}[A_p \ B_q \ C_r]$ must be less than m, because $[A_p \ B_q \ C_r]$ has m rows. Hence, there is a non-zero square matrix D of dimension m so that $DA_p = 0$, $DB_q = 0$ and $DC_r = 0$. Thus, we have

$$\begin{aligned}
A'(z) &\triangleq (I + Dz)A(z) = I + (A_1 + D)z + \cdots + (DA_{p-1} + A_p)z^p, \\
B'(z) &\triangleq (I + Dz)B(z) = B_1 z + \cdots + (DB_{q-1} + B_q)z^q, \\
C'(z) &\triangleq (I + Dz)C(z) = I + (C_1 + D)z + \cdots + (DC_{r-1} + A_r)z^r
\end{aligned}$$

and

$$(A'(z))^{-1}B'(z) \equiv (A(z))^{-1}B(z) \quad \text{and} \quad (A'(z))^{-1}C'(z) \equiv (A(z))^{-1}C(z),$$

which combining with Definition 2.1 implies that $A'(z) \equiv A(z)$, $B'(z) \equiv B(z)$ and $C'(z) \equiv C(z)$. In particular, $A_1 + D = A_1$, i.e. $D = 0$. The contradiction shows $\operatorname{rank}[A_p \ B_q \ C_r] = m$.

We now show the sufficiency. If $A(z)$, $B(z)$ and $C(z)$ in (2.1) have no common left factor and $\operatorname{rank}[A_p \ B_q \ C_r] = m$, then we can show that the system is identifiable. In fact, if the system were not identifiable, then, by Definition 2.1, there would exist three polynomial matrices $A'(z) = I + A'_1 z + \cdots + A'_{p'} z^{p'}$, $B'(z) = B'_1 z + \cdots + B'_{q'} z^{q'}$ and $C'(z) = I + C'_1 z + \cdots + C'_{r'} z^{r'}$ with $p' \le p$, $q' \le q$ and $r' \le r$ such that $[A(z) \ B(z) \ C(z)] \ne [A'(z) \ B'(z) \ C'(z)]$, but $(A'(z))^{-1}B'(z) \equiv (A(z))^{-1}B(z)$ and $(A'(z))^{-1}C'(z) \equiv (A(z))^{-1}C(z)$. Let $D(z) = A'(z)(A(z))^{-1}$. Then we have

$$[A'(z) \ B'(z) \ C'(z)] = D(z)[A(z) \ B(z) \ C(z)].$$

Since $A(z)$, $B(z)$ and $C(z)$ have no left common factor, there are three polynomial matrices $M_1(z)$, $M_2(z)$ and $M_3(z)$ such that

$$A(z)M_1(z)+B(z)M_2(z)+C(z)M_3(z)=I,$$

and hence $A'(z)M_1(z)+B'(z)M_2(z)+C'(z)M_3(z)=D(z)$, which implies that $D(z)=A'(z)(A(z))^{-1}$ is a polynomial matrix. Furthermore, since both $A'(z)$ and $A(z)$ have identity as their leading coefficient matrices, the leading coefficient matrix of $D(z)$ must be identity.

Set $D(z)=I+D_1z+\cdots+D_dz^d$ and $a=\max(p,q,r)$, and assume $A_i=0$ for $i>p$, $B_j=0$ for $j>q$ and $C_k=0$ for $k>r$. Then we have

$$\begin{aligned}
&[A'(z) \quad B'(z) \quad C'(z)]\\
=&[I \quad 0 \quad I]+[A_1+D_1 \quad B_1 \quad C_1+D_1]\,z\\
&+[A_2+D_2+D_1A_1 \quad B_2+D_1B_1 \quad C_2+D_2+D_1C_1]\,z^2\\
&+\cdots+[D_dA_a \quad D_dB_a \quad D_dC_a]\,z^{a+d}.
\end{aligned} \tag{2.10}$$

In the case $d\ge 1$ we must have $D_dA_p=0$, since $\deg A'(z)=p'\le p$. Similarly, we have $D_dB_q=0$, $D_dC_r=0$, and hence $D_d[A_p \ B_q \ C_r]=0$, which together with the fact that $\operatorname{rank}[A_p \ B_q \ C_r]=m$, implies that $D_d=0$. Suppose that $D_h=0$ for $h=k+1,\cdots,d$. If $k\ge 1$, then from (2.10) and $D_h=0$ $(h=k+1,\cdots,d)$ it follows that $D_k[A_p \ B_q \ C_r]=0$, which obviously implies that $D_k=0$. Therefore, we have $D(z)=I$, and hence $[A(z) \ B(z) \ C(z)]=[A'(z) \ B'(z) \ C'(z)]$ which contradicts $[A(z) \ B(z) \ C(z)]\neq[A'(z) \ B'(z) \ C'(z)]$.

The proof is completed. ∎

Theorem 2.2. If $A(z)$ is stable, $C^{-1}(z)-\frac{1}{2}I$ is strictly positive real and the system (2.1) is identifiable, then a strongly consistent estimate θ_t for θ can be given on the basis of input-output data of the system.

Proof. Let $\{v_t\}$ be a sequence of n-dimensional mutually independent random vectors with continuous distributions and satisfying

$$Ev_t=0,\quad Ev_tv_t^\tau=\frac{1}{t^\varepsilon}I,\quad \|v_t\|^2\le\frac{\sigma^2}{t^\varepsilon},\quad t\ge 1;\qquad v_t=0,\quad t\le 0, \tag{2.11}$$

$$\varepsilon\in\left[0,\ \frac{1}{2\mu+3}\right),\qquad \mu=(m+1)p+q+r,$$

where σ^2 is a fixed positive constant.

Take $u_t=v_t$ and estimate θ by θ_t:

$$\begin{aligned}
\theta_{t+1}=&\theta_t+a_tP_t\varphi_t(y_{t+1}^\tau-\varphi_t^\tau\theta_t),\\
P_{t+1}=&P_t-a_tP_t\varphi_t\varphi_t^\tau P_t,\qquad a_t=(1+\varphi_t^\tau P_t\varphi_t)^{-1},\\
\varphi_t^\tau=&[\,y_t^\tau\cdots y_{t-p+1}^\tau \quad u_t^\tau\cdots u_{t-q+1}^\tau \quad y_t^\tau-\varphi_{t-1}^\tau\theta_t\cdots y_{t-r+1}^\tau-\varphi_{t-r}^\tau\theta_{t-r+1}\,]
\end{aligned}$$

with $P_0=I$ and with θ_0 arbitrary.

Set

$$\begin{aligned}
\varphi_t^0=&[\,y_t^\tau\cdots y_{t-p+1}^\tau \quad u_t^\tau\cdots u_{t-q+1}^\tau \quad w_t^\tau\cdots w_{t-r+1}^\tau]^\tau\\
r_t^0=&mp+nq+mr+\sum_{i=0}^{t-1}\|\varphi_i^0\|^2
\end{aligned}$$

and denote by $\lambda_{\min}^{0}(t)$ the minimum eigenvalue of $I+\sum_{i=0}^{t-1}\varphi_i^0\varphi_i^{0\tau}$.

By Theorem 2 of [2] we know that

$$\|\theta-\theta_t\|^2=O\left(\frac{\log r_t^0(\log\log r_t^0)^c}{\lambda_{\min}^0(t)}\right),\qquad c>1.$$

From (2.7), (2.11) and stability of $A(z)$ it follows that $r_t^0=O(t)$, and hence

$$\|\theta-\theta_t\|^2=O\left(\frac{\log t(\log\log t)^c}{\lambda_{\min}^0(t)}\right),\qquad c>1.$$

Thus for the consistency of θ_t it suffices to show

$$\liminf_{t\to\infty} t^{-1+(\mu+1)\varepsilon}\lambda_{\min}^0(t)\neq 0\qquad \text{a.s.} \tag{2.12}$$

Set

$$f_t\triangleq(\det A(z))\theta_t^0,\qquad \det A(z)\triangleq a_0+a_1z+\cdots+a_sz^s,\qquad s\le mp.$$

By the Schwarz inequality and the fact that $\varphi_t^0=0$ for $t<0$, it is easy to see

$$\lambda_{\min}^f(t)=\inf_{\|x\|=1}\sum_{i=1}^t(x^\tau f_i)^2\le(s+1)\sum_{j=0}^s a_j^2\lambda_{\min}^0(t),$$

where $\lambda_{\min}^f(t)$ denotes the minimum eigenvalue of $\sum_{i=1}^t f_if_i^\tau$.

So for (2.12) it suffices to prove that

$$\liminf_{t\to\infty} t^{-1+(\mu+1)\varepsilon}\lambda_{\min}^f(t)\neq 0. \tag{2.13}$$

If this were not true, then there would exist a vector sequence $\{\eta_{t_k}\}$:

$$\eta_{t_k}=\left[\alpha_{t_k}^{(0)\tau}\cdots\alpha_{t_k}^{(p-1)\tau}\ \beta_{t_k}^{(0)\tau}\cdots\beta_{t_k}^{(q-1)\tau}\ \gamma_{t_k}^{(0)\tau}\cdots\gamma_{t_k}^{(r-1)\tau}\right]^\tau\in R^{mp+np+mr},$$

such that $\|\eta_{t_k}\|=1$ and

$$\liminf_{k\to\infty} t_k^{-1+(\mu+1)\varepsilon}\left(\sum_{i=1}^{t_k}(\eta_{t_k}^\tau f_i)^2\right)=0. \tag{2.14}$$

Let

$$\begin{aligned}H_{t_k}(z)&=\sum_{i=1}^{p-1}\alpha_{t_k}^{(i)\tau}z^i(\operatorname{Adj}A(z))\,[B(z)\ \ C(z)]+\sum_{i=1}^{q-1}\beta_{t_k}^{(i)\tau}z^i\left[(\det A(z))I_l\ \ 0\right]\\&\quad+\sum_{i=1}^{r-1}\gamma_{t_k}^{(i)\tau}z^i\,[0\ \ (\det A(z))I_m]\\&\triangleq\sum_{j=0}^{\mu}\left[h_{t_k}^{(j)\tau}\ \ g_{t_k}^{(j)\tau}\right]z^j,\qquad \mu\le\max(p,q,r)+mp-1,\end{aligned}$$

where $h_{t_k}^{(j)}$ and $g_{t_k}^{(j)}$ are n- and m-dimensional vectors respectively.

Applying the same argument as that used in (49)–(63) of [2], from (2.14) we conclude that

$$H_{t_k} \xrightarrow[k\to\infty]{} 0.$$

This means that there exists a unit vector

$$\eta^\tau = [\alpha_0^\tau \cdots \alpha_{p-1}^\tau \ \beta_0^\tau \cdots \beta_{q-1}^\tau \ \gamma_0^\tau \cdots \gamma_{r-1}^\tau], \quad \|\eta\| = 1$$

such that

$$\sum_{i=0}^{p-1} \alpha_i^\tau z^i (\mathrm{Adj}\, A(z)) B(z) = \sum_{i=0}^{q-1} \beta_i^\tau z^i (\det\, A(z)) I_n \tag{2.15}$$

and

$$\sum_{i=0}^{p-1} \alpha_i^\tau z^i (\mathrm{Adj}\, A(z)) C(z) = \sum_{i=0}^{r-1} \gamma_i^\tau z^i (\det\, A(z)) I_m. \tag{2.16}$$

Since $A(z)$, $B(z)$ and $C(z)$ have no common left factor, there are $M'(z)$, $N'(z)$ and $L'(z)$ such that

$$A(z)M'(z) + B(z)N'(z) + C(z)L'(z) = I, \tag{2.17}$$

which implies

$$\begin{aligned}\sum_{i=0}^{p-1} \alpha_i^\tau z^i (\mathrm{Adj}\, A(z)) &= \sum_{i=0}^{p-1} \alpha_i^\tau z^i (\mathrm{Adj}\, A(z)) \big(A(z)M'(z) + B(z)N'(z) + C(z)L'(z)\big) \\ &= \left(\sum_{i=0}^{p-1} \alpha_i^\tau z^i (\mathrm{Adj}\, A(z)) A(z)\right) M'(z) \\ &\quad + \left(\sum_{i=0}^{p-1} \alpha_i^\tau z^i (\mathrm{Adj}\, A(z)) B(z)\right) N'(z) \\ &\quad + \left(\sum_{i=0}^{p-1} \alpha_i^\tau z^i (\mathrm{Adj}\, A(z)) C(z)\right) L'(z).\end{aligned}$$

Therefore, by (2.15)–(2.16) we have

$$\begin{aligned}\sum_{i=0}^{p-1} \alpha_i^\tau z^i (\mathrm{Adj}\, A(z)) &= (\det\, A(z)) \left(\sum_{i=0}^{p-1} \alpha_i^\tau z^i M'(z) + \sum_{i=0}^{q-1} \beta_i^\tau z^i N'(z) + \sum_{i=0}^{r-1} \gamma_i^\tau z^i L'(z)\right) \\ &\triangleq (\det\, A(z)) \sum_{i=0}^{\lambda} \overline{\mu}_i^\tau z^i.\end{aligned} \tag{2.18}$$

Multiplying the equality (2.18) by $A(z)$, $B(z)$ and $C(z)$ from the right, we obtain respectively that

$$\sum_{i=0}^{p-1} \alpha_i^\tau z^i = \left(\sum_{i=0}^{\lambda} \overline{\mu}_i^\tau z^i\right) A(z), \qquad \sum_{i=0}^{q-1} \beta_i^\tau z^i = \left(\sum_{i=0}^{\lambda} \overline{\mu}_i^\tau z^i\right) B(z) \tag{2.19}$$

and

$$\sum_{i=0}^{r-1} \gamma_i^\tau z^i = \left(\sum_{i=0}^{\lambda} \overline{\mu}_i^\tau z^i\right) C(z). \tag{2.20}$$

Comparing coefficients for (2.19) and (2.20) and noticing that rank $[A_p \; B_q \; C_r] = m$ we find $\overline{\mu}_i = 0,\ i = 0, \cdots, \lambda$ and then from (2.19), (2.20) we conclude $\alpha_i = 0,\ \beta_j = 0,\ \gamma_k = 0,\ i = 0, \cdots, p-1;\ j = 0, \cdots, q-1;\ k = 0, \cdots, r-1$.

This contradicts $\|\eta\| = 1$ and at the same time verifies (2.12).

The proof of Theorem 2.2 is complete. ∎

Corollary 2.1. If the system noise in (2.1) is uncorrelated, i.e. $C(z) = I$, then the system is always identifiable whatever $A(z)$ and $B(z)$ are.

Remark 2.1. From this theorem it is seen that the results in [2]–[5] remain valid under weaker conditions, namely, the row-full-rank of A_p or B_q or C_r can be weakened to row-full-rank of $[A_p \, B_q \; C_r]$.

Remark 2.2. By using the recent result developed in [6], Theorem 2.2 remains true if stability of $A(z)$ is replaced by stability of $z^{-1}B(z)$.

The next theorem gives conditions different from those used in Theorem 2.2.

Theorem 2.3. If $m = n$, $z^{-1}B(z)$ is stable, system (2.1) is identifiable, and $C(z) - \frac{1}{2}I$ is strictly positive real, then a strongly consistent estimate θ_t for θ can be given on the basis of the input-output data of the system.

Proof. Let $\{v_t\}$ be a sequence of m-dimensional mutually independent random vectors with independent components having continuous distributions. Further, assume that

$$v_1 = 0, \quad Ev_t v_t^{\tau} = \frac{1}{\log^{\varepsilon} t} I, \quad \|v_t\|^2 \le \frac{\sigma^2}{\log^{\varepsilon} t}, \qquad \forall t \ge 2;$$

$$\varepsilon \in \left(0, \frac{1}{4s(m+2)}\right), \qquad s = \max(p, q, r+1),$$

where σ^2 is a constant.

Define θ_t' by the stochastic gradient algorithm

$$\begin{aligned} \theta_{t+1}' &= \theta_t' + \frac{1}{r_t'} \varphi_t' (y_{t+1}^{\tau} - \varphi_t'^{\tau} \theta_t'), \\ \varphi_t' &= \left[\, y_t^{\tau} \cdots y_{t-p+1}^{\tau} \;\; u_t^{\tau} \cdots u_{t-q+1}^{\tau} \;\; y_t^{\tau} - \varphi_{t-1}'^{\tau} \theta_{t-1}' \cdots y_{t-r+1}^{\tau} - \varphi_{t-r}'^{\tau} \theta_{t-r}' \,\right]^{\tau}, \\ r_t' &= mp + nq + mr + \sum_{i=0}^{t-1} \|\varphi_i'\|^2. \end{aligned}$$

It has been shown in [5] that at any time the estimate B_{1t}' given by θ_t' for B_1 is nondegenerate and

$$\sum_{i=0}^{t} (\|y_i\|^2 + \|u_i\|^2) = O(t) \qquad \text{a.s.,}$$

if the initial estimate B_{10}' for B_1 is nondegenerate and u_t is given by

$$u_t = u_t^0 + v_t$$

and

$$B_{1t}' u_t^0 = B_{1t}' u_t - \theta_t'^{\tau} \varphi_t'.$$

By Theorem 2 of [4], for the consistency of θ_t' it suffices to show that

$$\liminf_{t \to \infty} \frac{(\log t)^{\frac{1}{4} - \varepsilon}}{t} \lambda_{\min}^0(t) \ne 0. \tag{2.21}$$

Using the treatment used in Theorem 3 of [4], the assumption converse to (2.21) leads to (2.15) and (2.16), which imply a contradiction as is shown in Theorem 2.2.

Hence θ_t' is strongly consistent. ∎

Remark 2.3. It is clear, however, that Condition (2.6) cannot be satisfied by a deterministic system for which the analogues of Theorems 2.1–2.3 still take place. In this case Theorem 2.1 turns to the following statement. System (1.1) is identifiable if and only if $A(z)$ and $B(z)$ are left-coprime and rank $[A_p \quad B_q] = m$. Similarly, Theorems 2.2 and 2.3 remain true for deterministic systems if we remove conditions imposed on $C(z)$ in the theorems. This is because Theorem 3 of [2] and Theorem 3 of [5] are obviously true for deterministic systems if we remove conditions on $C(z)$ and $\{w_n\}$ in these theorems.

3. Conclusion

We have introduced a new definition of identifiability for multidimensional linear input-output systems and presented a necessary and sufficient condition for its satisfaction. In the case where the system is identifiable, by the methods given in this paper one can design a kind of experiment signal that leads to consistant estimates for the unknown coefficients of the system.

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